On products of Baire spaces

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APPLICATIONS
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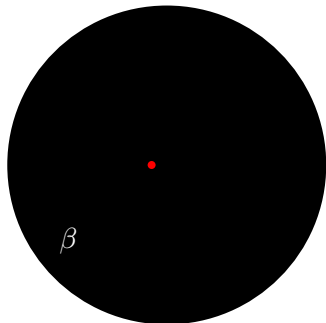
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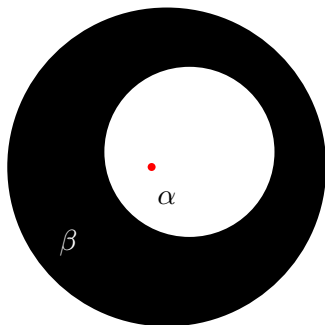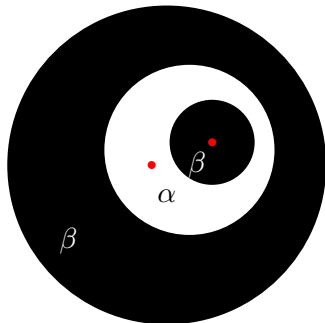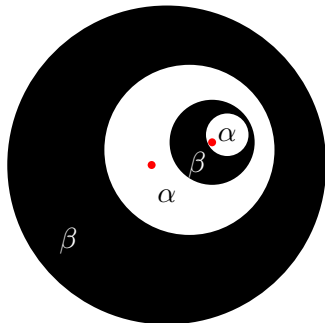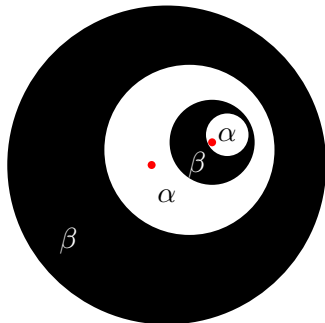
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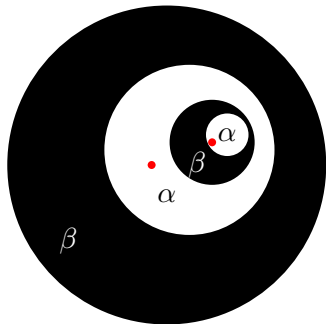
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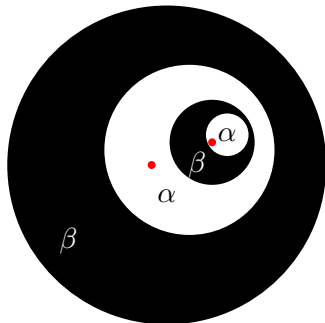
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α wins, if $\bigcap_n U_n \neq \emptyset$;

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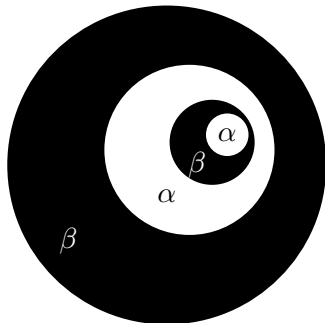α wins, if $\bigcap_n U_n \neq \emptyset$; β wins, if $\bigcap_n U_n = \emptyset$

Banach-Mazur game $BM(X)$

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Results about $BM(X), Ch(X)$

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Let X be a Baire space, Y_i have a dense set of W-points and $Ch(Y_i)$ be β-unfavorable for each $i \in I$. Then

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Let X be a Baire space, and Y_i a 1st countable, hereditarily Baire, R_0-space for each $i \in I$. Then

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